

3D ISING MODEL: THE SCALING EQUATION OF STATE

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ABSTRACT

The equation of state of the universality class of the 3D Ising model is determined numerically in the critical domain from quantum field theory and renormalization group techniques. The starting point is the five loop perturbative expansion of the effective potential (or free energy) in the framework of renormalized ϕ_3^4 field theory. The 3D perturbative expansion is summed, using a Borel transformation and a mapping based on large order behaviour results. It is known that the equation of state has parametric representations which incorporate in a simple way its scaling and regularity properties. We show that such a representation can be used to accurately determine it from the knowledge of the few first coefficients of the expansion for small magnetization. Revised values of amplitude ratios are deduced. Finally we compare the 3D values with the results obtained by the same method from the $\varepsilon = 4 - d$ expansion.

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1 Introduction

Renormalization group (RG) theory of second order phase transitions provides a complete description of all *universal* quantities (for a general reference on this article see for instance [1]). Among them the most studied are critical exponents, because they are easier to calculate, and because they have been used to test RG predictions by comparing them with other results (experiments, high or low temperature series expansion, Monte-Carlo simulations). For the $O(N)$ vector model, calculations are based upon the $(\phi^2)^2$ field theory

$$\mathcal{H}(\phi) = \int \left\{ \frac{1}{2} [\partial_\mu \phi(x)]^2 + \frac{1}{2} \lambda_2 \phi^2(x) + \frac{1}{4!} \lambda_4 [\phi^2(x)]^2 \right\} d^d x. \quad (1.1)$$

We recall that near the critical temperature T_c λ_2 is a linear measure of the temperature. If we denote by λ_{2c} the value for which the theory becomes massless ($T = T_c$) then the parameter t

$$t = \lambda_2 - \lambda_{2c} \propto T - T_c, \quad (1.2)$$

characterizes the deviation from the critical temperature.

To determine exponents two strategies have been used. One follows Parisi's suggestion [2] and is based on perturbation series calculated directly in three dimensions. Series up to six loops have been generated [3] which have been summed by summation methods based on a Borel transformation (first estimates for $N = 1$ were reported in [4]). The more accurate estimates are obtained after a conformal mapping which takes full advantage of the large order behaviour analysis [5,6]. For a review see [1,7]. In this article, as a test of the variant of the summation method we use, we have reexamined their determination.

Alternatively the $\varepsilon = 4 - d$ expansion known up to five loop order [8] has been summed by similar techniques [9] (note however that the authors used the slightly erroneous fifth order of Gorishny *et al*). Both expansions lead to consistent results. Moreover RG values for critical exponents have now resisted for many years to confrontation with lattice calculations and experimental determinations. Table 1 displays results for Ising-like systems ($N = 1$) coming from field theory under the form of summed $d = 3$ series and ε -expansion compared to experiments: binary fluids (*a*), liquid-vapour transitions (*b*) and antiferromagnets (*c*), see [7,10] for references. Only recent Helium superfluid transition experiments ($N = 2$) [11] now provide results which are consistent but more accurate than RG estimates. It would therefore be interesting to examine whether the series could now be extended to further decrease the apparent errors in $d = 3$ dimensions.

Other available estimates come from the analysis of High Temperature series in lattice models (table 2) and Monte-Carlo simulations. For the latter case let us quote two typical set of results in [19] the values $\nu = 0.629 \pm 0.003$ and $\eta =$

Table 1

Critical exponents for Ising-like systems: RG and experiments.

	γ	ν	β	α	$\theta = \omega\nu$
$d = 3$	1.2405 ± 0.0015	0.6300 ± 0.0015	0.325 ± 0.0015	0.110 ± 0.0045	0.50 ± 0.02
$\varepsilon - \text{exp.}$	1.2390 ± 0.0025	0.6310 ± 0.0015	0.327 ± 0.0015	0.110 ± 0.0045	0.51 ± 0.03
(a)	1.236 ± 0.008	0.625 ± 0.010	0.325 ± 0.005	0.112 ± 0.005	0.50 ± 0.03
(b)	$1.23 - 1.25$	0.625 ± 0.006	$0.316 - 0.327$	0.107 ± 0.006	0.50 ± 0.03
(c)	1.25 ± 0.01	0.64 ± 0.01	0.328 ± 0.009	0.112 ± 0.007	

Table 2

Critical exponents for Ising-like systems: HT series.

	γ	ν	α	$\theta = \omega\nu$
[12]	1.239 ± 0.002	0.631 ± 0.003		
[13]	1.2385 ± 0.0025	0.6305 ± 0.0015		0.57 ± 0.07
[14]	1.239 ± 0.003	0.631 ± 0.004		
[15]	1.2395 ± 0.0004	0.632 ± 0.001	0.105 ± 0.007	0.54 ± 0.05
[16]	1.239 ± 0.003	0.632 ± 0.003	$0.101 \pm .004$	
[17]	1.237 ± 0.002	0.630 ± 0.0015		0.52 ± 0.03
[18]			0.104 ± 0.004	

0.027 ± 0.005 are reported. In [20] one finds $\nu = 0.625 \pm 0.001$ and $\eta = .025 \pm 0.006$ while θ varies in the range 0.44–0.53.

Other universal quantities, like the equation of state [21,22,23], and some amplitude ratios [24,23,25], have also been calculated but the estimates are less accurate because the series are shorter. Perturbative calculations are more difficult, in particular for $N \neq 1$ (continuous symmetries), due to the presence of two different masses (transverse and longitudinal) in an external magnetic field and Goldstone singularities on the coexistence curve ($H = 0$, $T < T_c$). Probably also less effort have been invested up to now.

In this article we therefore present a determination of the equation of state for the $N = 1$, $d = 3$ case, where longer series are available. Our calculations are based on perturbative expansion at fixed $d = 3$ dimension [2]. Five loop series for the renormalized effective potential of the ϕ_3^4 theory have been first reported by Bagnuls *et al.* [26], but the printed tables contain some serious misprints. These have been noticed by Halfkann and Dohm who have published corrected values [27] (for detailed explanations see our appendix). Finally full analytic results up

to three loops have recently become available [28].

The main technical difficulty that one faces in $d = 3$ calculations is how to continuously extrapolate field theory results from $T > T_c$ to $T < T_c$. Indeed because the massless (or critical) theory is IR divergent in perturbation theory at any fixed dimension $d < 4$, calculations can be performed only in the massive phase (in contrast with the ε -expansion). In [29] one method is suggested which has then been used [30] (see also [26]) to predict some amplitude ratios. We present in this article a different approach that extends a suggestion in [1], and is motivated by the simplicity of the parametric representation [31] of the equation of state within the framework of the ε -expansion [22,23].

The first step of our approach is a summation of the available perturbative series to obtain a non-perturbative determination of small field expansion of the effective potential (coefficients of $\phi^6, \phi^8, \phi^{10}$) of the continuous ϕ_3^4 field theory at the IR fixed point. This result is interesting in itself in view of the recent effort devoted to the problem (Monte Carlo lattice simulations, High Temperature series and approximate numerical solutions of Exact Renormalization Group). With the help of a parametric representation we are then able to reconstruct the full scaling equation of state, which is therefore the main result of our article. We then use it to calculate several amplitude ratios which have been considered before in the literature. We compare our predictions with other available results (ε -expansion, High Temperature Expansion, Monte Carlo, Exact Renormalization Group and of course experiments).

The set-up of our article is thus the following. In section 2 we briefly review the properties of the equation of state and the definitions of amplitude ratios. In section 3 we recall known results about the ε -expansion, while in section 4 the idea of parametric equation of state and its application to the ε -expansion are presented. In section 5 we explain the method we have used in our $d = 3$ calculations. In section 6 the summation of the perturbative expansions for the effective potential is discussed. The results concerning the effective potential and the (parametric) equation of state are reported in section 7, while section 8 contains our results for amplitude ratios and some concluding remarks.

2 Effective action and equation of state

In this article the general framework is the massive theory renormalized at zero momentum. The correlation functions $\Gamma_r^{(n)}$ of the renormalized field $\phi_r = \phi/\sqrt{Z}$ are fixed by the normalization conditions

$$\Gamma_r^{(2)}(p; m, g) = m^2 + p^2 + O(p^4), \quad (2.1a)$$

$$\Gamma_r^{(4)}(p_i = 0; m, g) = m^{4-d}g. \quad (2.1b)$$

In this renormalization scheme one trades the initial parameters λ_2 and λ_4 for g and m . The renormalized coupling constant g is dimensionless. It has to be set to

its IR fixed point value g^* . In [6] the value $g^* = 23.73 \pm 0.09$ has been numerically estimated from the series published in [3]. The mass parameter m is proportional to the physical mass, or inverse correlation length, of the high temperature phase. It behaves for $t \propto T - T_c \rightarrow 0_+$ as $m \propto t^\nu$, where ν is the correlation length exponent (see [1] for details). Therefore the parameter m is singular at T_c and thus the extrapolation to the low temperature phase is non-trivial. Within the framework of the ε -expansion instead, it is possible to first construct the massless theory and then the theory in the full critical domain as an expansion in powers of the deviation t from the critical temperature. Equivalently the problem can be solved because scaling laws are exactly satisfied. In the summed $d = 3$ perturbative expansion, at the IR fixed point g^* , scaling relations are satisfied only within summation errors. The parametric representation of the equation of state [31] will provide a solution to this problem.

From the conditions (2.1) it follows that the free energy \mathcal{F} per unit volume expressed in terms of the “renormalized” magnetization φ , i.e. the expectation value of the renormalized field $\phi_r = \phi/\sqrt{Z}$, has a small φ expansion of the form (in d dimensions)

$$\mathcal{F}(\varphi) = \Gamma(\varphi)/\text{vol.} = \mathcal{F}(0) + \frac{1}{2}m^2\varphi^2 + \frac{1}{4!}m^{4-d}g\varphi^4 + O(\varphi^6), \quad (2.2)$$

where $\Gamma(\varphi)$ is also the generating functional of One Particle Irreducible correlators restricted to constant fields, or the effective potential of the renormalized theory.

The equation of state is the relation between the magnetic field H , the magnetization $M = \langle \phi \rangle$ (the “bare” field expectation value) and the temperature which is represented by the parameter t (eq. (1.2))

$$H = \frac{\partial \mathcal{F}}{\partial M}.$$

Near the critical point the equation of state has Widom’s scaling form

$$H(M, t) = M^\delta f(t/M^{1/\beta}). \quad (2.3)$$

It is convenient to introduce the rescaled variable z

$$\varphi = m^{(d-2)/2} z / \sqrt{g}, \quad (2.4)$$

and set

$$\mathcal{F}(\varphi) - \mathcal{F}(0) = \frac{m^d}{g} V(z, g). \quad (2.5)$$

The critical behaviour is described by the IR fixed point, but we keep here the notation g to remind that we must first sum up perturbative expansion for V and then take $g = g^*$.

The equation of state is obtained from the derivative F of the reduced effective potential V with respect to z

$$F(z, g) = \frac{\partial V(z, g)}{\partial z}. \quad (2.6)$$

Ising-like symmetry implies that F has an expansion of the form

$$F(z, g) = z + \frac{1}{6}z^3 + \sum_{l=2} F_{2l+1}(g)z^{2l+1}. \quad (2.7)$$

From eqs. (2.4,2.3) we conclude

$$z \propto Mt^{-\beta}. \quad (2.8)$$

Comparing then the coefficient of M in the small M expansion of $H(M, t)$ (see also eq. (2.13))

$$H(M, t) = H_0 t^\gamma M + O(M^3), \quad (2.9)$$

where H_0 is a numerical constant, with the small z expansion of the function $F(z)$ we conclude

$$H \propto t^{\beta\delta} F(z), \quad (2.10)$$

where the relation between exponents

$$\gamma = \beta(\delta - 1), \quad (2.11)$$

has been used. One property of the function $H(M, t)$ which plays an essential role in our analysis is *Griffith's analyticity*: it is regular at $t = 0$ for $M > 0$ fixed, and simultaneously it is regular at $M = 0$ for $t > 0$ fixed.

Amplitude ratios. Universal amplitude ratios are numbers characterizing the behaviour of thermodynamical quantities near T_c in the critical domain, which in addition do not depend on the normalizations of magnetic field, magnetization and temperature. Several amplitude ratios commonly considered in the literature can be derived from the scaling equation of state:

The specific heat. The singular part of the specific heat, i.e. the ϕ^2 2-point correlation function at zero momentum, behaves like

$$C_H = A^\pm |t|^{-\alpha}, \quad t \propto T - T_c \rightarrow \pm 0. \quad (2.12)$$

The ratio A^+/A^- then is universal.

The magnetic susceptibility. The magnetic susceptibility χ in zero field, i.e. the ϕ 2-point function at zero momentum, diverges like

$$\chi = C^\pm |t|^{-\gamma}, \quad t \rightarrow \pm 0. \quad (2.13)$$

The ratio C^+/C^- then is also universal.

Other amplitude ratios. On the critical isotherm the magnetic susceptibility behaves as

$$\chi = C^c/H^{1-1/\delta}; \quad (2.14)$$

the spontaneous magnetization vanishes as

$$M = B(-t)^\beta. \quad (2.15)$$

One can then define the following universal ratio:

$$R_c = \alpha A^+ C^+ / B^2, \quad (2.16)$$

which corresponds to the relation between exponents

$$\alpha + 2\beta + \gamma = 2.$$

Indeed using this relation we verify that R_c is proportional to $F(0, t)M^{-2}\chi$ which is normalization independent. Aharony and Hohenberg define the following universal combination

$$R_\chi = B^{\delta-1}C^+/(C^c\delta)^\delta,$$

which corresponds to the relation (2.11). It is related to the quantity Q_1 defined by Fisher and Tarko, $R_\chi = Q_1^{-\delta}$.

3 The ε -expansion

Let us first recall the results concerning exponents and the equation of state which have been obtained within the framework of the $\varepsilon = 4 - d$ expansion.

3.1 Critical exponents

Although the RG functions of the ϕ^4 theory and therefore the critical exponents are known up to five-loop order [8], we give the expansions only up to the order needed in this article, i.e. to three loops, referring to the literature for details. The zero $g^*(\varepsilon)$ of the β -function is $g^*(\varepsilon) = 16\pi^2\varepsilon/3 + O(\varepsilon^2)$. The values of the critical exponents γ , β and $\delta = 1 + \gamma/\beta$ are

$$\begin{aligned} \gamma &= 1 + \frac{1}{6}\varepsilon + \frac{25}{324}\varepsilon^2 + \left(\frac{701}{17496} - \frac{2}{27}\zeta(3)\right)\varepsilon^3 + O(\varepsilon^4) \\ 2\beta &= 1 - \frac{1}{3}\varepsilon + \frac{1}{81}\varepsilon^2 + \left(\frac{163}{8748} - \frac{2}{27}\zeta(3)\right)\varepsilon^3 + O(\varepsilon^4) \\ \delta &= 3\left(1 + \frac{1}{3}\varepsilon + \frac{25}{162}\varepsilon^2 + \frac{539}{8748}\varepsilon^3\right) + O(\varepsilon^4), \end{aligned}$$

in which $\zeta(s)$ is the Riemann ζ -function. (At orders four and five $\zeta(5)$ and $\zeta(7)$ successively appear. It has also been shown that starting at six loop order new non- ζ like numbers will appear [32].) Other exponents can be obtained from scaling relations.

3.2 The scaling equation of state

The ε -expansion of the scaling equation of state (eq. (2.3)) has been determined up to order ε^2 for the general $O(N)$ model, and order ε^3 for $N = 1$. Adjusting the normalizations of H and M to simplify the analytic expressions one finds at order ε^2 :

$$f(x) = 1 + x + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \varepsilon^3 f_3(x) + O(\varepsilon^4), \quad (3.1)$$

with:

$$\begin{aligned} f_1(x) &= \frac{1}{6}(x+3)L \\ f_2(x) &= \frac{1}{72}(x+9)L^2 + \frac{25}{324}(x+3)L \\ f_3(x) &= \frac{L^3(27+x)}{1296} + \frac{L^2(675+246x+25x^2)}{1944(3+x)} \\ &\quad + \frac{L(1617+701x-1296x\zeta(3))}{17496} + \frac{x^2(-1-2\lambda+4\zeta(3))}{108(3+x)} \end{aligned}$$

with

$$L = \log(x+3),$$

while λ is a constant

$$\lambda = \frac{1}{3}\psi'(1/3) - \frac{2}{9}\pi^2 = 1.17195361934\dots \quad (3.2)$$

The expression (3.1) is not valid for x large, i.e. for small magnetization M . In this regime the magnetic field H has a regular expansion in odd powers of M , i.e. in the variable $z \propto x^{-\beta}$. Substituting in eq. (3.1) $x = x_0 z^{-1/\beta}$ (the constant x_0 takes care of the normalization of z) and expanding in ε one finds at order ε^3 for the function (2.6)

$$F(z) = \tilde{F}_0(z) + \varepsilon \tilde{F}_1(z) + \varepsilon^2 \tilde{F}_2(z) + \varepsilon^3 \tilde{F}_3(z),$$

with

$$\tilde{F}_0 = z + \frac{1}{6}z^3 \quad (3.3a)$$

$$\begin{aligned} \tilde{F}_1 &= \frac{1}{12}(-z^3 + \tilde{L}(2z + z^3)) \\ \tilde{F}_2 &= \frac{1}{1296}(-50z^3 + \tilde{L}(100z - 4z^3) + \tilde{L}^2(18z + 27z^3)) \end{aligned} \quad (3.3b)$$

$$\begin{aligned} \tilde{F}_3 &= \frac{1}{69984} \frac{1}{(2+z^2)} \left[\tilde{L}^3(108z + 540z^3 + 243z^5) + \tilde{L}^2(1800z + 1494z^3 + 621z^5) \right. \\ &\quad \left. + \tilde{L}(-1622z^5 + z^3(1504 - 5184\zeta(3)) + z(5608 - 10368\zeta(3))) \right. \\ &\quad \left. + z^5(-997 - 648\lambda + 3888\zeta(3)) + z^3(-2804 + 5184\zeta(3)) \right] \end{aligned} \quad (3.3c)$$

and

$$\tilde{L} = \log(1 + z^2/2).$$

While within the framework of the formal ε -expansion one can easily pass from one expansion to the other one, still a matching problem arises if one wants to apply the ε -expansion for $d = 3$, i.e. $\varepsilon = 1$. One is thus naturally led to look for a uniform representation of the equation of state valid in both limits. Josephson-Schofield parametric representation has this property.

4 Parametric representation of the equation of state

We parametrize M and t in terms of two variables R and θ , setting :

$$\begin{cases} M = m_0 R^\beta \theta, \\ t = R(1 - \theta^2), \\ H = h_0 R^{\beta\delta} h(\theta), \end{cases} \quad (4.1)$$

where h_0, m_0 are two normalization constants [31]. This parametrization also corresponds in terms of the scaling variables x of eq. (3.1) or z from eq. (2.4) to set

$$z = \rho\theta/(1 - \theta^2)^\beta, \quad \theta > 0, \quad (4.2)$$

$$x = x_0 \rho^{-1/\beta} (1 - \theta^2) \theta^{-1/\beta}, \quad (4.3)$$

where ρ is some other positive constant.

Then the function $h(\theta)$ is an odd function of θ regular near $\theta = 1$, which is x small, and near $\theta = 0$ which is x large. We choose h_0 such that

$$h(\theta) = \theta + O(\theta^3).$$

The function $h(\theta)$ vanishes for $\theta = \theta_0$ which corresponds to the coexistence curve $H = 0, T < T_c$. From θ_0 we obtain the universal rescaled spontaneous magnetization $|z_0|$

$$|z_0| = \rho\theta_0/(\theta_0^2 - 1)^\beta. \quad (4.4)$$

Note that the mapping (4.2) is not invertible for values of θ such that $z'(\theta) = 0$. One verifies that the derivative vanishes for $\theta^2 = 1/(1 - 2\beta) \sim 2.86$. We shall see that this value is reasonably larger than the largest value of $\theta^2 = \theta_0^2$ we will have to consider.

Finally it is useful for later purpose to write more explicitly the relation between the function $F(z)$ of eq. (2.6) and the function $h(\theta)$:

$$h(\theta) = \rho^{-1} (1 - \theta^2)^{\beta\delta} F(z(\theta)). \quad (4.5)$$

Expanding both functions

$$F(z) = z + \frac{1}{6}z^3 + \sum_{l=2} F_{2l+1} z^{2l+1}, \quad (4.6)$$

$$h(\theta)/\theta = 1 + \sum_{l=1} h_{2l+1} \theta^{2l}, \quad (4.7)$$

we find the relations

$$h_3 = \frac{1}{6}\rho^2 - \gamma \quad (4.8a)$$

$$h_5 = \frac{1}{2}\gamma(\gamma - 1) + \frac{1}{6}(2\beta - \gamma)\rho^2 + F_5\rho^4 \quad (4.8b)$$

$$h_7 = \frac{1}{6}\gamma(\gamma - 1)(\gamma - 2) + \frac{1}{12}(2\beta - \gamma)(2\beta - \gamma + 1)\rho^2 \\ + (4\beta - \gamma)F_5\rho^4 + F_7\rho^6 \quad (4.8c)$$

...

From the parametric representation of the equation of state it is also possible to derive a representation for the singular part of the free energy per unit volume. Setting:

$$\mathcal{F}_{\text{sg.}}(M, t) \equiv \Gamma_{\text{sg.}}(M, t)/\text{vol.} = h_0 m_0 R^{2-\alpha} g(\theta), \quad (4.9)$$

one finds for $g(\theta)$ a differential equation:

$$h(\theta) (1 - \theta^2 + 2\beta\theta^2) = 2(2 - \alpha)\theta g(\theta) + (1 - \theta^2) g'(\theta). \quad (4.10)$$

The integration constant is fixed by requiring the regularity of $g(\theta)$ at $\theta = 1$. In the same way the inverse magnetic susceptibility is given by:

$$\chi^{-1} = (h_0/m_0)R^\gamma g_2(\theta), \quad (4.11)$$

with:

$$g_2(\theta) (1 - \theta^2 + 2\beta\theta^2) = 2\beta\delta\theta h(\theta) + (1 - \theta^2) h'(\theta). \quad (4.12)$$

Note that g_2 then in general has a pole at $\theta^2 = 1/(1 - 2\beta)$, the point at which the mapping (4.2) is not invertible.

The functions $g(\theta)$ and $g_2(\theta)$ will be used to calculate several universal ratios of amplitudes.

4.1 Amplitude ratios

The amplitude ratios defined in section 2 are related to the scaling equation of state. They can be calculated from the functions $h(\theta)$, $g(\theta)$ and $g_2(\theta)$ appearing in the parametric representation, and thus ultimately from the function $h(\theta)$ alone (eqs. (4.10,4.12)). One verifies that indeed they do not depend on the variable R and the constants m_0, h_0 appearing in (4.1).

The magnetic susceptibility. The magnetic susceptibility in zero field can be calculated from the function $g_2(\theta)$ defined by equation (4.12). One obtains (equation (4.12)):

$$\frac{C^+}{C^-} = (\theta_0^2 - 1)^{-\gamma} \frac{h'(\theta_0)(1 - \theta_0^2)}{h'(0)(1 - \theta_0^2 + 2\beta\theta_0^2)}. \quad (4.13)$$

The specific heat. This ratio is directly related to the function $g(\theta)$ defined by equation (4.10):

$$\frac{A^+}{A^-} = (\theta_0^2 - 1)^{2-\alpha} \frac{g(0)}{g(\theta_0)}. \quad (4.14)$$

Other universal ratios. Similarly the ratios R_c and R_χ can be derived from the functions $g(\theta)$ and $h(\theta)$

$$R_c = -\alpha(1-\alpha)(2-\alpha) \frac{g(0)(\theta_0^2 - 1)^{2\beta}}{h'(0)\theta_0^2}, \quad (4.15)$$

$$R_\chi = \frac{h(1)\theta_0^{\delta-1}}{h'(0)(\theta_0^2 - 1)^\gamma}. \quad (4.16)$$

4.2 Parametric representation in the ε -expansion

Up to order ε^2 the constant m_0 (or ρ) can be chosen in such a way that the function $h(\theta)$ reduces to:

$$h(\theta) = \theta \left(1 - \frac{2}{3}\theta^2\right) + O(\varepsilon^2). \quad (4.17)$$

The simple model in which $h(\theta)$ is approximated by a cubic odd function of θ is called the linear parametric model. At order ε^2 the linear parametric model is exact, but at order ε^3 the introduction of a term proportional to θ^5 becomes necessary [22,23]. One finds:

$$h(\theta) = \theta(1 + h_3\theta^2 + h_5\theta^4) + O(\varepsilon^4), \quad (4.19)$$

with

$$h_3 = -\frac{2}{3} \left(1 + \frac{\varepsilon^2}{12}\right), \quad h_5 = \frac{\varepsilon^3}{27} \left(\zeta(3) - \frac{1}{2}\lambda - \frac{1}{4}\right), \quad (4.20)$$

where λ is the constant (3.2).

The function $h(\theta)$ vanishes on the coexistence curve for $\theta = \theta_0$:

$$\theta_0^2 = \frac{3}{2} \left(1 - \frac{\varepsilon^2}{12}\right) + O(\varepsilon^3). \quad (4.21)$$

Note that h_3 and thus θ_0 are determined only up to order ε^2 . It follows

$$\rho^2 = 6(\gamma + h_3) = 2 \left(1 + \frac{1}{2}\varepsilon + \frac{7}{108}\varepsilon^2\right) = 3.13 \pm 0.13,$$

because h_3 is determined only up to order ε^2 , and

$$\begin{aligned} |z_0| &= \rho\theta_0(\theta_0^2 - 1)^{-\beta} \\ &= \sqrt{3} \times 2^\beta \left[1 + \frac{1}{4}\varepsilon + \frac{73}{864}\varepsilon^2 + \left(\frac{1}{24}\lambda - \frac{7}{36}\zeta(3) + \frac{5581}{93312}\right)\varepsilon^3\right] \sim 2.87 \pm 0.06 \end{aligned}$$

(in this case we summed by a Padé [1, 2]).

Remark. Even in the more general $O(N)$ case, the parametric representation automatically satisfies the different requirements about the regularity properties of the equation of state and leads to uniform approximations. However for $N > 1$ the function $h(\theta)$ still has a singularity on the coexistence curve, due to the presence of Goldstone modes in the ordered phase and has therefore a more complicated form. The nature of this singularity can be obtained from the study of the non-linear σ -model. It is not clear whether a simple polynomial approximation would be useful. For $N = 1$ instead, one expects at most an essential singularity on the coexistence curve, due to barrier penetration, which is much weaker and non-perturbative in the small ε - or small g -expansion.

Amplitude ratios. The use of eqs. (4.14–4.16) together with the expression of $h(\theta)$ eqs. (4.19,4.20) evaluated at $\varepsilon = 1$ gives us the predictions for the amplitude ratios reported in table 5 (ε -expansion (b)). Moreover from these equations the known ε -expansion of various amplitude ratios, [24,23,25], can be easily obtained. The ratio C^+/C^- , related to the magnetic susceptibility in zero field, is

$$\frac{C^+}{C^-} = \frac{2^{\gamma+1}}{6\beta-1} \left[1 + \left(\frac{2\lambda+1}{4} - \zeta(3) \right) \frac{\varepsilon^3}{12} \right] + O(\varepsilon^4) \quad (4.22a)$$

$$= 2^{\gamma-1}(\delta-1) \left[1 + \frac{1}{36} \left(\zeta(3) + \frac{3}{2}\lambda + \frac{1}{4} \right) \varepsilon^3 \right] \quad (4.22b)$$

$$= 2^\gamma \left[1 + \frac{1}{2}\varepsilon + \frac{25}{108}\varepsilon^2 + \left(\frac{1159}{11664} + \frac{1}{36}\zeta(3) + \frac{1}{24}\lambda \right) \varepsilon^3 \right]. \quad (4.22c)$$

The ratio C^+/C^- can be expressed at order ε^2 entirely in terms of critical exponents. This form follows naturally from the parametric representation of the equation of state. The ε^3 relative correction is of the order of only 3%. The three first expressions yield for $\varepsilon = 1$ respectively (the exponents being replaced by the central values of the summed ε -expansion):

$$\frac{C^+}{C^-} = 4.688, 4.757, 4.863.$$

The spread gives for this short series an indication about the uncertainty about the result.

The ratio A^+/A^- at order ε^2 is given by

$$\frac{A^+}{A^-} = 2^{\alpha-2} \left[1 + \varepsilon + \left(\frac{43}{54} - \frac{1}{6}\lambda - \zeta(3) \right) \varepsilon^2 \right] + O(\varepsilon^3).$$

The ε -expansion of R_c is:

$$R_c = \frac{1}{9} 2^{-2\beta-1} \varepsilon \left[1 + \frac{17}{27}\varepsilon + \left(\frac{989}{2916} - \frac{4}{9}\zeta(3) - \frac{2}{9}\lambda \right) \varepsilon^2 \right] + O(\varepsilon^4). \quad (4.23)$$

R_χ is given by

$$R_\chi = 3^{(\delta-3)/2} 2^{\gamma+(1-\delta)/2} \left[1 + \left(\frac{1}{72} + \frac{1}{36}\lambda - \frac{1}{18}\zeta(3) \right) \varepsilon^3 \right] + O(\varepsilon^4). \quad (4.24)$$

5 The perturbative expansion at fixed dimension three

Critical exponents and several other universal quantities have been estimated using the perturbative expansion at fixed dimension $d < 4$. Since the massless theory is then IR divergent, calculations have been performed within the framework of the 3D perturbative expansion renormalized at zero momentum for the massive ϕ^4 field theory. As in the case of the ε -expansion, it is necessary to first determine the IR stable zero g^* of the function $\beta(g)$ which is given by a few terms of a divergent expansion. The obvious problem is that we have no longer a small parameter in which to expand and g^* is a number of order 1. Already at this point a summation method is required. Note also that at any finite order the results for universal quantities become renormalization scheme dependent in contrast with the results of the ε -expansion.

On the other hand, because one-loop diagrams have, in 3 dimensions, a simple analytic expression, it has been possible to extend the calculation of RG functions in the N -vector model up to six-loop order. The expansions are [3,33]:

$$\begin{aligned} \beta(\tilde{g}) = & -\tilde{g} + \tilde{g}^2 - \frac{308}{729}\tilde{g}^3 + 0.3510695978\tilde{g}^4 - 0.3765268283\tilde{g}^5 \\ & + 0.49554751\tilde{g}^6 - 0.749689\tilde{g}^7 + O(\tilde{g}^8), \end{aligned} \quad (5.1)$$

$$\begin{aligned} \gamma^{-1}(\tilde{g}) = & 1 - \frac{1}{6}\tilde{g} + \frac{1}{27}\tilde{g}^2 - 0.0230696213\tilde{g}^3 + 0.0198868203\tilde{g}^4 \\ & - 0.02245952\tilde{g}^5 + 0.0303679\tilde{g}^6, \end{aligned} \quad (5.2)$$

$$\begin{aligned} \eta(\tilde{g}) = & \frac{8}{729}\tilde{g}^2 + 0.0009142223\tilde{g}^3 + 0.0017962229\tilde{g}^4 \\ & - 0.00065370\tilde{g}^5 + 0.0013878\tilde{g}^6, \end{aligned} \quad (5.3)$$

with the normalization:

$$\tilde{g} = 3g/(16\pi). \quad (5.4)$$

It has thus been possible to determine critical exponents quite accurately (table 1).

To determine the equation of state or universal ratios of amplitudes a new problem arises. In this framework it is more difficult to calculate physical quantities in the ordered phase because the theory is parametrized in terms of the disordered phase correlation length $\xi = m^{-1}$ which is singular at T_c (as well as correlation function normalization condition, eq.(2.1)). Let us consider the perturbative expansion of the scaling equation of state (2.6). For example at one-loop order for $d = 3$ the function $F(z, g)$ is given by:

$$\begin{aligned} F(z, g) = & z + \frac{1}{6}z^3 - \frac{1}{8\pi}gz \left[(1 + z^2/2)^{1/2} - 1 - z^2/4 \right] \\ = & z + \frac{1}{6}z^3 + \frac{1}{256\pi}gz^5 - \frac{1}{2^{13}\pi}gz^7 + O(z^9), \end{aligned} \quad (5.5)$$

where the subtractions, due to the mass and coupling normalizations, are determined by the conditions (2.1). This expression is adequate for the description

of the disordered phase, but all terms in the loopwise expansion become singular when t goes to zero for fixed magnetization, that corresponds to the limit $m \rightarrow 0$, φ fixed for the effective potential, or $z \rightarrow \infty$ for reduced variables (see eq.(2.4)). In this regime we know from eq.(2.3) (or from RG considerations) that the equation of state is

$$H(M, t = 0) \propto M^\delta \Rightarrow F(z) \propto z^\delta, \quad (5.6)$$

exactly.

In the case of the ε -expansion the very essence of the method is that one is doing perturbative expansions for the theory *at* the critical point: it follows that the scaling relations (and thus the limit eq. (5.6)) are exactly satisfied order by order. Moreover the change to the variable $x \propto z^{-1/\beta}$ (more appropriate for the regime $t \rightarrow 0$) gives an expression for $f(x) \propto F(x^{-\beta})x^{\beta\delta}$ that is explicitly regular in $x = 0$ (Griffith's analyticity at the critical point): the singular powers of $\log x$ induced by the change of variables cancel non trivially at each order, leaving only regular corrections.

The situation changes when one deals with the perturbation theory at $d = 3$ dimensions: for g generic the system is no more at the critical point, and consequently scaling properties are not satisfied order by order in g . In particular the change to the Widom function $f(x)$ will introduce the singular terms in $\log x$ that violates Griffith's analyticity. An analogous problem arises if one first sums the series at $g = g^*$ before changing to the variable x . In this case the singular contributions (in the form of powers of x) do not cancel, as a result of unavoidable numerical summation errors.

Several approaches can be used to deal with the problem of reaching the ordered phase. In [29] a method to calculate amplitude ratios is proposed which has been also used in [26,30].

If we are concerned only with amplitude ratios another strategy is available that in some sense bypasses the problem. Near the critical point physical quantities have simple power law singularities in t . To reach the coexistence curve, i.e. $t < 0$, $H = 0$, it is possible to proceed by analytic continuation in the complex t -plane. From equations (2.8,2.10)

$$H(M, t) \propto t^{\beta\delta} F(z), \quad z \propto Mt^{-\beta},$$

and the knowledge that at M fixed $H(M, t)$ is regular at T_c or $t = 0$ we conclude that $t < 0$ corresponds to z complex

$$t = |t| e^{i\pi}, \Rightarrow z = |z| e^{-i\pi\beta}. \quad (5.7)$$

The scaling variable $H(-t)^{-\beta\delta}$ is then given by:

$$H(-t)^{-\beta\delta} = e^{i\pi\beta\delta} F(z) = |F(z)|. \quad (5.8)$$

Finally the spontaneous magnetization is given in terms of the complex zero z_0 of $F(z)$.

It is in particular possible to evaluate ratios of amplitudes of singularities above and below T_c : we calculate the complex zero $z_0(g)$ of $F(z, g)$ and substitute it in other quantities. The result is complex but its absolute value converges towards the correct result. For example the ratio of amplitudes for the magnetic susceptibility (eq. (2.10)) is given by:

$$C^+/C^- = e^{-i\pi\gamma} F'(z_0(g), g) / F'(0, g) = |F'(z_0(g), g)|. \quad (5.9)$$

We thus get a series expansion in g for C^+/C^- which can be summed at $g = g^*$ with techniques described in section 6. However this method does not allow to calculate for t small and thus is not well suited to determine the full equation of state. Moreover if we want to sum perturbation series with the method recalled in section 6, we will face the problem that the large order behaviour depends on the value of the variable z itself.

Encouraged by the results obtained within the ε -expansion scheme, we develop a different, more powerful strategy, based on the parametric representation.

The parametric representation. Order dependent mapping (ODM). The problem that we face is the following: to reach the ordered region $t < 0$ for the (summed) equation of state function $F(z, g^*)$, we must cross the point $z = \infty$ for which the exact behaviour is dictated by eq. (5.6). The idea that the ε -expansion suggests is to introduce a new field variable θ and an auxiliary function $h(\theta)$ defined as in (4.2,4.5): in this way the exact function $h(\theta)$ will be regular near $\theta = 1$ (i.e. $z = \infty$) and up to the coexistence curve. However, the approximate $h(\theta)$ that we obtain by summing perturbation theory at fixed dimension, will still not be regular. The singular terms generated by the mapping eq. (4.2) at $\theta = 1$ will not cancel exactly due to summation errors. The last step we propose is to Taylor expand the approximate expression of $h(\theta)$ around $\theta = 0$ and to truncate the expansion, enforcing in this way regularity. The next question then is to which order should we expand? Since the coefficients of the θ expansion are in one to one correspondence with the coefficients of small z expansion of the function $F(z, g^*)$ the maximal power of θ in $h(\theta)$, should be equal to the maximal power of z which can be reasonably well summed. Indeed although the small z expansion of $F(z)$ at each finite loop order in g contains an infinite number of terms, the determination of the coefficients of the higher powers of z is increasingly difficult. The reasons are twofold:

(i) The number of terms of the series in g required to get an accurate estimate of F_l increases with l (see section 6).

(ii) At any finite order in g the function $F(z)$ has spurious singularities in the complex z plane (see e.g. eq. (5.5), $z^2 = -2$) that dominate the behaviour of the coefficients F_l for l large.

In view of these difficulties we have to ensure the fastest possible convergence of the small θ expansion. For this purpose we use the freedom in the choice

of the arbitrary parameter ρ in eq.(4.2): we determine it to minimize the last term in the truncated small θ expansion, i.e. increasing the importance of small powers of theta which are more accurately determined. This is nothing but the application to this particular example of the series summation method based on ODM [34].

This strategy applied to the available data, leads at leading order for $h(\theta)$ to a polynomial of degree 5, whose coefficients are given by the relations (4.8):

$$h(\theta) = \theta[1 + h_3(\rho)\theta^2 + h_5(\rho)\theta^4]. \quad (5.10)$$

For the range of admissible values for F_5 the coefficient h_5 of θ^5 given by eq. (4.8b) has no real zero in ρ . It has a minimum instead

$$\rho^2 = \rho_5^2 = \frac{1}{12F_5}(\gamma - 2\beta). \quad (5.11)$$

Substituting this value of ρ into expression (5.10) we obtain the first approximation for $h(\theta)$. At next order we look for a minimum ρ_7 of $|h_7(\rho)|$. We find a polynomial either of degree 5 in θ , when h_7 has a real zero, or of degree 7 when it has only a minimum.

We have not explored beyond $h_9(\rho)$ because already F_9 is too poorly determined.

Finally we want to point out that, to compute the amplitude ratios and thus to reach the ordered phase quantities we implement in the present framework the idea of analytic continuation outlined in the previous subsection. In particular the ordered phase at zero magnetic field will correspond to the choice $\theta = \theta_0$ where θ_0 will be the zero of $h(\theta)$ closest to the origin.

6 The equation of state: series summation

Our first task is thus to determine the coefficients F_{2l+1} as accurately as possible. To sum the perturbation series the Borel–Leroy [35] transformation has been used, combined with a conformal mapping [36] (a simplified version of method used in [9] for critical exponents). Let $\mathcal{S}(g)$ be the function whose series has to be summed. We then transform the series:

$$\mathcal{S}(g) = \sum_{k=0} \mathcal{S}_k g^k, \quad (6.1)$$

into:

$$\mathcal{S}(g) = \sum_{k=0} B_k(b) \int_0^\infty t^b e^{-t} u^k(gt) dt, \quad (6.2)$$

with:

$$u(s) = \frac{\sqrt{1+as}-1}{\sqrt{1+as}+1}. \quad (6.3)$$

The coefficients B_k are calculated by expanding in powers of g the r.h.s. of equation (6.2) and identifying with expansion (6.1). (For motivations and details see e.g. [1]). The constant a (corresponding to the scaled variable $\tilde{g} = \frac{3}{16\pi}g$ of [3]) has been determined by the large order behaviour analysis [1],

$$a(d=3) = 0.147774232, \quad (6.4)$$

for the perturbative expansion in $d=3$ dimensions. The parameter b is adjusted empirically to improve the convergence of the transformed series: for example one looks for the intersection between the results at two consecutive orders in k (or the minimum of the difference), or a point of least sensitivity. Moreover the value of b has to stay in a reasonable range around the value predicted by the large order behaviour. It is also to be expected that the summation method will be efficient if the coefficients \mathcal{S}_k are already approaching the asymptotic large order regime. This is what we are going to test first. The expectations are that for the coefficients F_{2l+1} the situation will deteriorate with increasing l : indeed the large order behaviour estimate has increasingly large corrections. Moreover for l large F_{2l+1} becomes dominated by the perturbative singularity of $F(z)$ at $z^2 = -2$ while the summed function has singularities at different locations.

Large order behaviour analysis [37]. The series are taken from [26,27,28]. The coefficients of the perturbative expansion of

$$F_l \equiv \frac{1}{l!} g^{1-(l+1)/2} m^{(l+1)/2-3} \Gamma^{(l+1)}(p_i=0, m, g) = \sum_k F_{lk} g^k,$$

have the large order behaviour:

$$F_{lk} \underset{k \rightarrow \infty}{\sim} \frac{1}{l!} C_{l+1} \left(-\frac{2}{3I_4}\right)^{k-1+(l+1)/2} \Gamma(k+l+3/2)$$

$$C_l \equiv -\frac{e^{-9I_4/(32\pi)}}{\pi(2\pi)^{3/2}} \left(\frac{4I_1^2}{I_4}\right)^{l/2} \left[\bar{D}_1 \left(\frac{\frac{3}{4}I_4}{I_6 - I_4}\right)^3\right]^{-1/2}.$$

The constant $I_4 = \frac{9a}{32\pi}$ ($a \equiv a(d=3)$ is given by eq. (6.4)), is related to the action of the instanton which describes the instability of the ϕ^4 for negative coupling, while $I_1 = 31.691522$, $I_6 = 659.868352$, (related to integrals of power 1, 6 of the scaled instanton solution) and $\bar{D}_1 = 10.544$ (related to the functional determinant) have been computed in [38].

Table 3 shows that, as anticipated, the asymptotic regime sets in later when l increases. We thus expect that the efficiency of the summation will correspondingly decrease, and indeed this is what happens.

Summation. Following an idea introduced in [9] for the summation of the ε -expansion we have in addition made a homographic transformation on the coupling constant g to displace some singularities in the complex g -plane:

$$g = g'/(1 + pg'/g^*). \quad (6.5)$$

Table 3
Large Order Behaviour Analysis : $F_{lk}/F_{lk}^{\text{as}}$.

l	$k = 2$	$k = 3$	$k = 4$	$k = 5$
5	1.9646	1.5894	1.4226	1.3498
7	1.9680	1.9965	1.9603	1.9204
9	1.4443	1.9879	2.229	2.333

We have looked for a value of the parameters p and b for which the results were specially insensitive to the order k : in practice we minimized the absolute differences of results corresponding to three successive orders. Finally to verify that our method gives results consistent with those of the previous analysis [6] we have applied it to the RG β -function and exponents. Figure 1 gives the last four orders for the β -function as a function of the parameter b and at the minimal value $p = 0.196$. One observes that the curves flatten with increasing order, as expected. The central value slightly differs from the result given in [6]:

$$g^* = 23.70 \pm 0.05 , \quad \text{with } \omega = \beta'(g^*) = 0.79 \pm 0.01 ,$$

and the apparent error is smaller. Note that a variation of g^* of 0.05 yields a variation of 0.04 of the β -function.

Figure 2 displays the results for the exponent γ for the four last orders, for $g^* = 23.70$, and for the minimal value $p = .182$. One obtains

$$\gamma = 1.2405 \pm 0.0012 .$$

With the same method one finds for the exponent β

$$\beta = 0.3250 \pm 0.0015 .$$

One notices the prefect agreement with the result of table 1, although the central value for g^* has changed. The reason is that the central values given in [6] for the exponents largely rely on the pseudo- ε expansion, which avoids the explicit determination of g^* .

7 Numerical results: Equation of state

We first calculate the first coefficients of the small z expansion of the equation of state function $F(z)$ (or equivalently of the reduced effective potential, eq. (2.6), summed at the I.R. fixed point g^*). In figure 3 we display the behaviour of F_5 in terms of b at the minimal value of the parameter $p = .182$ (see section 6 for definition of parameters).

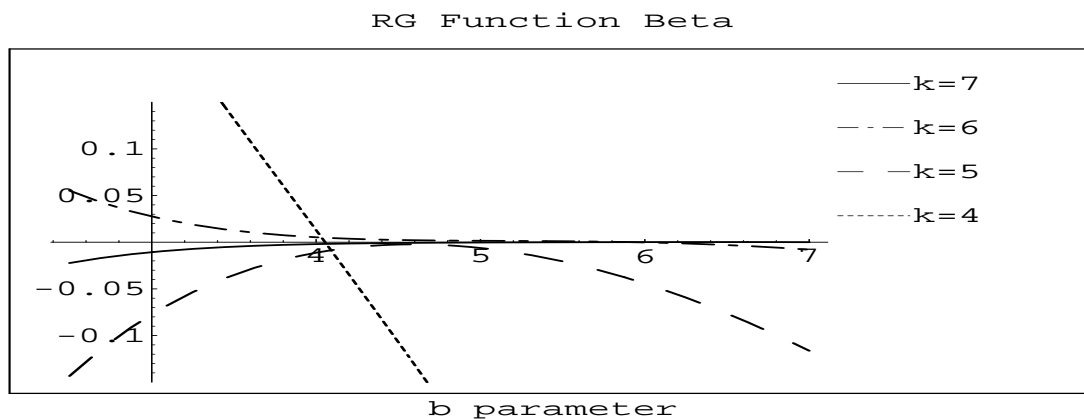


Fig. 1 The RG β -function plotted vs. the parameter b for successive orders k .

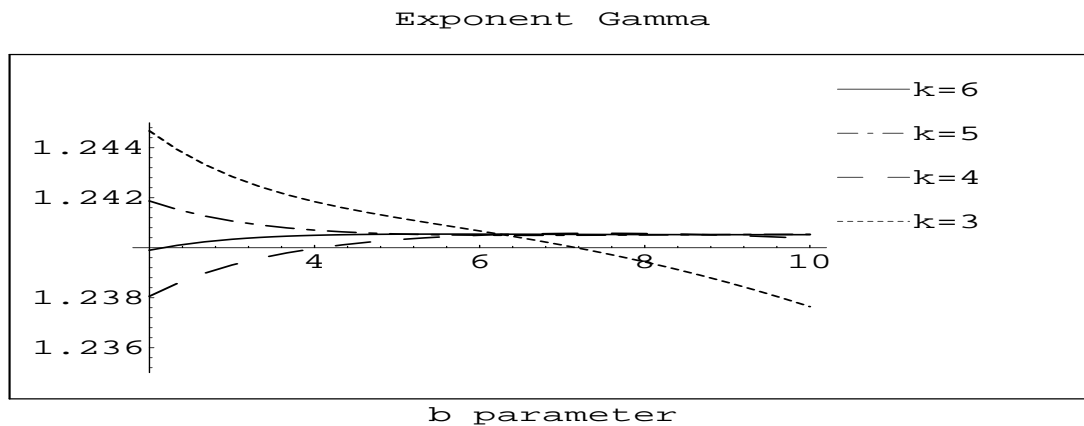


Fig. 2 The exponent γ plotted vs. the parameter b for successive orders k .

Table 4 contains our results together with other published estimates of the coefficients of the small z expansion of $F(z)$. For what concerns the ε -expansion, we have taken the ε -expansion of $h(\theta)$, eqs. (4.19,4.20) setting $\varepsilon = 1$. We have then calculated the coefficients F_{2l+1} , using the numerical values of the more

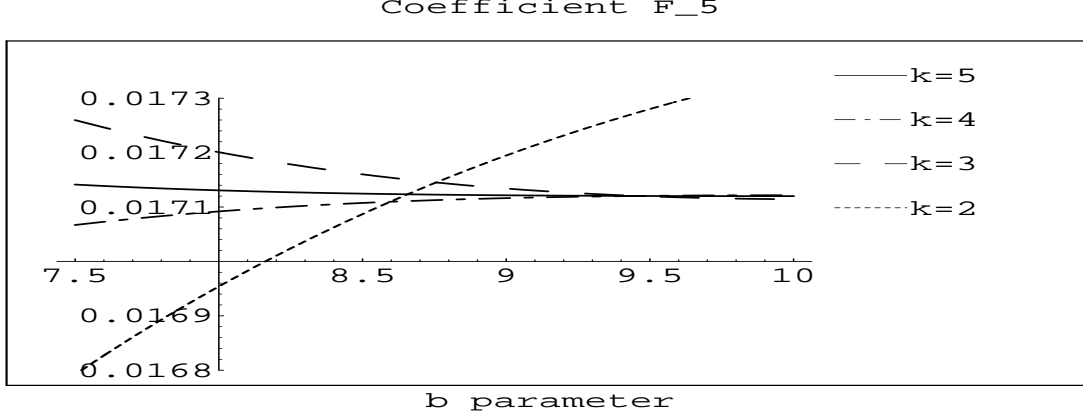


Fig. 3 The coefficient F_5 plotted vs. the parameter b for successive orders k .

accurately determined exponents γ, β (second line of table 1). This procedure has led to much stabler results for F_{2l+1} than a direct summation of the ε -expansion. The quoted errors are nevertheless only indicative (and subjective) because $h(\theta)$ is known only to order ε^2 for h_3 and ε^3 for h_5 . Note that although g^* is known only to order ε^2 [40]

$$g^*(\varepsilon) = \frac{1}{3}\Gamma(d/2)(4\pi)^{d/2}\varepsilon \left(1 + \frac{61}{54}\varepsilon\right) + O(\varepsilon^3),$$

evaluation of the expression at $\varepsilon = 1$ gives nevertheless a number of the right order of magnitude $g^* = 28$.

Parametric representation. We then determine by the ODM method the coefficient ρ and the function $h(\theta)$, as explained in section 5. We obtain successive approximations in the form of polynomials of increasing degree for $h(\theta)$. Note that we here have a simple test of the relevance of the ODM method. Indeed, once $h(\theta)$ is determined, assuming the values of the critical exponents γ and β , we can recover a function $F(z)$ which has an expansion to all orders in z . As a result we obtain a prediction for the coefficients F_{2l+1} which have not yet been taken into account to determine $h(\theta)$. The relative difference between the predicted value and the calculated one gives an idea about the accuracy of the ODM method. Indeed from the values $F_5 = 0.01712, \gamma = 1.2405, \beta = 0.3250$, we obtain

$$F_7 = 4.94 \times 10^{-4}, \quad F_9 = -3.2 \times 10^{-5}, \quad F_{11} = 8.5 \times 10^{-8} \dots$$

Table 4
Equation of state.

	g^*	F_5	F_7	F_9
ε -exp.	28.	0.0176 ± 0.0004	$(4.5 \pm 0.3) \times 10^{-4}$	$-(3.2 \pm 0.2) \times 10^{-5}$
$d = 3$	23.70 ± 0.05	0.01712 ± 0.00006	$(4.96 \pm 0.49) \times 10^{-4}$	$-(6.2 \pm 4.0) \times 10^{-5}$
$d = 3$ [41]	23.73	.01727	.0010	
HT [42]	23.72 ± 1.49	0.0205 ± 0.0052		
HT [39]	24.45 ± 0.15	$.017974 \pm .00015$		
HT [43]	23.69	.0166	.00055	.00001
MC [44]	23.3 ± 0.5	0.0227 ± 0.0026		
MC [45]	$24.5 \pm .2$	0.027 ± 0.002	$0.00236 \pm .00040$	
ERG [46]	28.9	0.016	4.3×10^{-4}	
ERG [47]	20.7 ± 0.2	0.0173 ± 0.0001	$(5.0 \pm 0.2) \times 10^{-4}$	$-(4. \pm 2.) \times 10^{-5}$

We note that the value for F_7 is very close to the central value we find by direct series summation, while the value for F_9 is within the error bars. This result gives us confidence in our method. It also shows that the value of F_9 obtained by direct summation contains very little new information, it provides only a consistency check. Therefore the simplest representation of the equation of state, consistent with all data, is given by

$$h(\theta)/\theta = 1 - 0.76147(36) \theta^2 + 7.74(11) \times 10^{-3} \theta^4, \quad (7.1)$$

(errors on the last digits in parentheses) that is obtained from $\rho^2 = 2.8743$ (fixed according eq. (5.11)). This expression of $h(\theta)$ has a zero at

$$\theta_0^2 = 1.331, \quad (7.2)$$

to which corresponds the value of the complex root z_0 of $F(z)$, $|z_0| = 2.801$ (the phase, given by eq. (5.7), is $-i\pi\beta$). The coefficient of θ^7 in eq. (7.1) is smaller than 10^{-3} . Note that for the largest value of θ^2 which corresponds to θ_0^2 , the θ^4 term is still a small correction. Finally note that the corresponding values for the ε -expansion are $h_3 = -0.72$, $h_5 = 0.013$. These values are reasonably consistent between them, since a small change in the value of ρ with a correlated change in h_3 induces a very small change in physical quantities.

The Widom scaling function $f(x)$, eq. (2.3), (with $f(-1) = 0$ and $f(0) = 1$) can easily be derived by (numerically) solving the following system:

$$\begin{cases} f(x) = \theta^{-\delta} h(\theta)/h(1) \\ x = \left(\frac{1 - \theta^2}{1 - \theta_0^2} \right) \left(\frac{\theta_0}{\theta} \right)^{1/\beta} \end{cases} \quad (7.3)$$

Our results are displayed in fig. 4, where they are compared with ε^2 (Linear Parametric Model) and ε^3 predictions, High Temperature series ("quintic" fit amplitude ratios results of [48] and Padé approximants of bcc lattice results of [49]) as well with Exact Renormalization Group (a fit reported in [50] that approximates numerical data up to 1%). Note that the rapid increase of relative errors for $x < 0$ is not very significative since the absolute values are small.

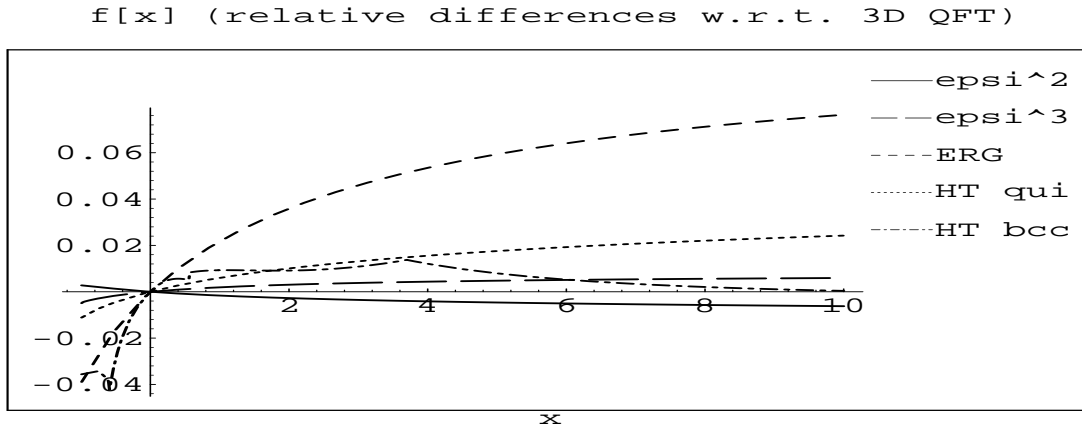


Fig. 4 Widom scaling function $f(x)$: relative errors with respect 3D QFT plotted vs. x .

One notices the general agreement between different predictions within 6% up to $x = 10$. In particular the general agreement between our predictions and the ε -expansion is striking. The main disagreement with other predictions comes from the region $x \rightarrow \infty$, i.e. from the small magnetization region, where our predictions should be specially reliable. Finally let us notice that the results [48,49] are based some obsolete estimates of critical amplitudes ($\gamma = 5/4$, $\beta = 5/16$).

8 Amplitude ratios. Conclusion

We finally use the expressions (4.14–4.16) with the critical exponents determined by earlier work (from longer series) to calculate various amplitude ratios. As a comparison we apply the same procedure to the ε -expansion at order ε^3 (using the values of exponents determined from the ε -expansion at order ε^5). It is necessary to again stress that as in table 4 the errors we then quote are largely tentative since the order ε^3 is the first order where some reliable results may be expected.

Table 5 contains a comparison of amplitude ratios as obtained from RG, lattice calculations and experiments on binary mixtures, liquid–vapour, uniaxial magnetic systems.

Table 5
Amplitude ratios.

	A^+/A^-	C^+/C^-	R_c	R_χ
ε – exp. (a)	0.524 ± 0.010	4.9		1.67
ε – exp. (b)	0.547 ± 0.021	4.70 ± 0.10	0.0585 ± 0.0020	1.649 ± 0.021
$d = 3$ fixed (a)	0.541 ± 0.014	4.77 ± 0.30	0.0594 ± 0.001	1.7
$d = 3$ fixed (b)	0.536 ± 0.019	4.82 ± 0.10	0.0576 ± 0.0020	1.674 ± 0.019
HT series	0.523 ± 0.009	4.95 ± 0.15	0.0581 ± 0.0010	1.75
bin. mix.	0.56 ± 0.02	4.3 ± 0.3	0.050 ± 0.015	1.75 ± 0.30
liqu. – vap.	0.48–0.53	4.8–5.2	0.047 ± 0.010	1.69 ± 0.14
magn. syst.	0.49–0.54	4.9 ± 0.5		

Results for ε -expansion (a) are taken from [25] and [23] (direct Padé summation of each corresponding series), while (b) are our results, obtained by first summing $h(\theta)$ and then computing ratios by use of eqs. (4.13–4.16). The results $d = 3$ fixed dimension (a) are taken from [26] and refers to direct summation up to $O(g^5)$ while $d = 3$ (b) is the present work. High Temperature results are taken from [51] (R_χ from [52]). Experimental data are extracted from [53], to which we refer for an updated and wider list of results (and references). We note the general consistency of the results obtained by different methods. In addition in table 7 we compare our predictions with the results of [39] concerning the following the universal ratios:

$$R_0 = \frac{(C_4^+)^2}{C^+ C_6^+}, \quad R_3 = -\frac{C_3^- B}{(C^-)^2}, \quad \frac{C_4^+}{C_4^-}$$

where the C_k^\pm are defined by $\partial^k \chi / \partial H^k = C_{k+2}^\pm |t|^{-\gamma-k\beta\delta}$ and the other quantities have been defined in section 2. After some algebra one finds the relation

$$R_0 = \frac{1}{10}(1 - 12F_5)^{-1}$$

that has thus been used in Table 4.

Conclusion. Working within the framework of renormalized quantum field theory and renormalization group, we have shown that the presently available five

Table 6
Other amplitude ratios.

	R_0	R_3	C_4^+/C_4^-
HT series [39]	0.1275 ± 0.0003	6.4 ± 0.2	-9.0 ± 0.3
d=3 fixed	0.12586 ± 0.00012	6.10 ± 0.06	-9.2 ± 0.6
ε -expansion	0.1268 ± 0.0008	6.11 ± 0.10	-8.3 ± 1.0

loop series allow, after proper summation, to determine with reasonable accuracy the complete scaling equation of state for 3D Ising-like systems. The parametric representation of the equation of state plays a central role in our analysis. As a consequence new estimates of some amplitude ratios have been obtained. Clearly a similar strategy could be applied to other quantities in a magnetic field, in the scaling region. We want to stress that an extension of the ε -expansion of the equation of state for $N = 1$ to order ε^4 or even better ε^5 , that does not seem an impossible task, would significantly improve the ε -expansion estimates and would therefore be extremely useful. Moreover the present approach could be extended to systems in the universality class of the $(\phi^2)_3^2$ field theory for higher N , provided expansions of the renormalized effective potential at high enough order are computed.

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APPENDICES

A1 Perturbative expansion of the effective potential

Table 7
Coefficients F_{blk} .

b	l	$k = 0$	$k = 1$	$k = 2$
1	0	$-1/(12\pi)$		
2	0	$1/(128\pi^2)$		
2	1		$-1/(96\pi^2)$	
3	0	$(5 + 8 \log \frac{3}{4})/(6144\pi^3)$	$1/(768\pi^3)$	
3	1	$k_3/(6912\pi^3)$		
3	2	$k_4/(20736\pi^3)$		
4	0	$0.449205291 \times 10^{-6}$	$0.122593698 \times 10^{-5}$	
4	1	$-.115567060 \times 10^{-5}$	$-1/(18432\pi^4)$	
4	2	$0.100772569 \times 10^{-5}$		
4	3	$-.300586776 \times 10^{-6}$		
5	0	$-.451638912 \times 10^{-7}$	$(3 - 8 \log \frac{3}{4})/(1179648\pi^5)$	$-1/(294912\pi^5)$
5	1	$0.806878091 \times 10^{-7}$	$k_3/(1327104\pi^5)$	
5	2	$-.100671980 \times 10^{-6}$	$k_4/(1327104\pi^5)$	
5	3	$0.647972392 \times 10^{-7}$		
5	4	$-.165599191 \times 10^{-7}$		

Our starting point is the renormalized five loop effective potential in Minimal Subtraction ($\overline{\text{MS}}$) scheme, that can be written in the form:

$$\begin{aligned}
V_{\overline{\text{MS}}}(\phi, m_s, \lambda_4, \bar{\mu}) &\equiv \frac{m_s^2}{2} \phi^2 + \frac{\lambda_4}{4!} \phi^4 \\
&+ M^3 \sum_{b=1}^5 \sum_{l=1}^{b-1} \sum_{k=0}^2 F_{blk} \left(\frac{\lambda_4}{M} \right)^{b-1} \left(\frac{\frac{\lambda_4}{2} \phi^2}{M^2} \right)^l \left(\log \frac{k_1 \bar{\mu}}{M} \right)^k \quad (A1.1)
\end{aligned}$$

where m_s (λ_4) are the renormalized mass (coupling), and we defined $M^2 \equiv m_s^2 + \frac{\lambda_4}{2} \phi^2$, $k_1 \equiv \sqrt{e}/3$. The constant F_{blk} are reported in Table 7.

Analytical results up to three loops have been obtained by [28], and can be parametrized by the two constants

$$\begin{aligned} k_3 &\equiv -\frac{9}{2} + \frac{9\pi^2}{4} - \frac{27}{2}(\log(4/3))^2 - 27\text{Li}_2(1/4) \\ &= 9.36271548728022813982 \dots \end{aligned}$$

$$\begin{aligned} k_4 &\equiv -\frac{27\pi^2}{8} + \frac{81}{4}(\log(4/3))^2 + 54\log(4/3) + \frac{81}{2}\text{Li}_2(1/4) \\ &\quad - \frac{54}{\sqrt{2}} \int_0^1 \frac{dx}{\sqrt{3-x^2}} \left(\log \frac{3}{4} + \log \frac{3+x}{2+x} - \frac{x^2}{4-x^2} \log \frac{4}{2+x} + \frac{x}{2+x} \log \frac{3+x}{3} \right) \\ &= -6.43307044049269064141 \dots \end{aligned}$$

These expressions involve the dilogarithm or Spence function

$$\text{Li}_2(z) = \int_z^0 \frac{\log(1-t)}{t} dt .$$

We have obtained the analytic expressions of some four and five loop coefficients by imposing the $\overline{\text{MS}}$ RG equation on the effective potential, that in our notations reads:

$$\left(\bar{\mu} \frac{\partial}{\partial \bar{\mu}} + \frac{1}{12(4\pi)^2} \frac{\lambda_4^2}{m_s} \frac{\partial}{\partial m_s} \right) V_{\overline{\text{MS}}} = 0. \quad (\text{A1.2})$$

We want to emphasize that eq. (A1.2) above holds to all orders in this scheme because mass divergences arise only at two loop order ($\lambda_2 = m_s^2 + \frac{1}{12(4\pi)^2} \frac{\lambda_4^2}{\epsilon}$) and no additional finite renormalization is needed in the $\overline{\text{MS}}$ scheme. In practice (A1.2) should be satisfied at b loops up to order $O(\lambda_4^b)$ (if we take $\phi \sim O(1/\sqrt{\lambda_4})$): the term λ_4^2 enforces relations between the coefficients of \log , \log^2 at order b and terms at order $b-2$. The constant $F_{4,0,1}$ cannot be fixed because the corresponding relation implied by (A1.2) involves terms without field dependence (i.e. “cosmological terms”) that are not considered in (A1.1).

Numerical results up to five loops for F_{blk} have been first published in Table II of [26] (derived from Table I that contains the values of single diagrams, see also [3]). Some misprints in Table I, of eq. (3.2) and Table II of [26] are reported in [27], where Table II is rederived from the corrected values of Table I, a corrected eq. (3.2) and eq. (B1–B3) of [26]. We checked once more Table 1 of [27], and we obtained the same values, provided that $-(-A_d)^b$ of eq. (3.2) of [26] is replaced by $(-A_d)^b$ (a misprint is present in the prescriptions of [27]), provided that the formulae eqs. (B1–B3) of [26] are intended to hold for the dimensionless Γ_b functions (i.e. divided by a factor g_0^3 compared to definition eq. (3.2)), and that \tilde{X}_0 is considered as dependent from \tilde{r}_0 when taking derivatives $\frac{\partial}{\partial \tilde{r}_0}$ as in eq. (B3).

For our purposes we are interested in the physical mass scheme, that is fixed by normalization conditions (2.1). The required finite renormalization can be summarized by the following relation:

$$V_M(\phi, m, g) = V_{\text{MS}}(\sqrt{Z}\phi, Z_m m, Z_g m g, \bar{\mu}) \quad (\text{A1.3})$$

that defines the effective potential in the new scheme in terms of the new renormalized mass m and dimensionless coupling g .

The renormalization constant Z is fixed by (2.1) and we rewrite here for completeness. It is obtained from the derivative of the two-point function:

$$Z^{-1} = \left(\frac{\partial \Gamma_{\text{MS}}^{(2)}}{\partial p^2} / \Gamma_{\text{MS}}^{(2)} \right)_{p=0} = \left(\frac{\partial \Gamma_0^{(2)}}{\partial p^2} / \Gamma_0^{(2)} \right)_{p=0}.$$

The last equality is due to the absence of field renormalization between bare and $\overline{\text{MS}}$ quantities. Thus the constant Z is the same that one would obtain in passing from bare quantities to mass scheme and can be obtained from the known RG functions of the latter scheme:

$$\log Z = \int_0^g dg' \frac{\eta(g')}{\beta(g')}.$$

The other constant can then be fixed by

$$\begin{aligned} \frac{d^2 V_M}{d\phi^2}(0, m, g) &= m^2 \\ \frac{d^4 V_M}{d\phi^4}(0, m, g) &= m g. \end{aligned}$$

Due to the fact that the potential in the mass scheme will not depend on $\bar{\mu}$, one can simplify tedious calculations by choosing $\bar{\mu} \propto m_s$.

The reduced potential V is then obtained from:

$$V(z, g) \equiv \frac{m^3}{g} V_M(z\sqrt{m/g}, m, g).$$

We report here the final (numerical) results for the coefficients F_l of the Taylor expansion in z of $F(z, g)$, the derivative with respect to z of V (see (2.6)).

$$\begin{aligned} F_5 &= +\frac{g}{256\pi} - \frac{g^2}{2048\pi^2} + \frac{g^3(-274 + 18k_3 - 24k_4 + 27\log(\frac{3}{4}))}{442368\pi^3} \\ &\quad - 2.25018021 \times 10^{-7} g^4 + 1.977252 \times 10^{-8} g^5 \\ F_7 &= -\frac{g}{1024\pi} + \frac{65g^2}{147456\pi^2} + \frac{g^3(646 - 40k_3 + 80k_4 - 45\log(\frac{3}{4}))}{1769472\pi^3} \\ &\quad + 4.17878698 \times 10^{-7} g^4 - 4.512966 \times 10^{-8} g^5 \\ F_9 &= +\frac{5g}{16384\pi} - \frac{25g^2}{98304\pi^2} + \frac{g^3(-69099 + 4200k_3 - 11200k_4 + 3780\log(\frac{3}{4}))}{339738624\pi^3} \\ &\quad - 5.21924201 \times 10^{-7} g^4 + 6.986358 \times 10^{-8} g^5 \end{aligned}$$

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